TM 5 Pr 3.11

DERIVE THE EXPRESSIONS FOR THE ENERGY AND ENERGY LOSS CURVES (FIG 3.8) FOR THE DAMPED OSCILLATOR AND REPRODUCE THEM WITH MATHEMATICA.

FOR THE UNDERDAMPED OSCILLATOR

$$x(t) = Ae^{-\beta t}\cos(\omega_s t - \delta)$$

$$\dot{x}(t) = -\omega_s A e^{-\beta t}\left[\sin(\omega_s t - \delta) + \frac{\beta}{\omega_s}\cos(\omega_s t - \delta)\right]$$

THE ENERGY IS [REPLACING $\theta = (\omega_s t - \delta)$]

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}A^2 e^{-2\beta t}\left\{k\cos^2\theta + m\omega_s^2\left[\sin\theta + \frac{\beta}{\omega_s}\cos\theta\right]^2\right\}$$

$$= \frac{1}{2}A^2 e^{-2\beta t}\left\{k\cos^2\theta + m\omega_s^2\left[\sin^2\theta + \frac{2\beta}{\omega_s}\sin\theta\cos\theta + \frac{\beta^2}{\omega_s^2}\cos^2\theta\right]\right\}$$

NOTING THAT $\omega_N^2 = \frac{k}{m}$ AND $\omega_s^2 = \omega_N^2 - \beta^2$

$$E = \frac{m}{2}A^2 e^{-2\beta t}\left\{\omega_N^2\cos^2\theta + (\omega_N^2 - \beta^2)\sin^2\theta + 2\beta\omega_s\sin\theta\cos\theta + \beta^2\cos^2\theta\right\}$$

NOTING $\cos^2\theta + \sin^2\theta = 1$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$ AND $\sin 2\theta = 2\sin\theta\cos\theta$

$$E = \frac{m}{2}A^2 e^{-2\beta t}\left\{\omega_N^2 + \beta^2\cos 2\theta + \beta\omega_s\sin 2\theta\right\}$$

REPLACING $\theta = \omega_s t - \delta$

$$\boxed{E = \frac{m}{2}A^2 e^{-2\beta t}\left\{\omega_N^2 + \beta^2\cos 2(\omega_s t - \delta) + \beta\sqrt{\omega_N^2 - \beta^2}\sin 2(\omega_s t - \delta)\right\}}$$

QED!

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THE RATE OF ENERGY LOSS IS THE TIME DERIVATIVE,

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{mA^2}{2} e^{-2\beta t}\left\{\omega_N^2 + \beta^2 \cos 2(\omega_s t - \delta) + \beta\omega_s \sin 2(\omega_s t - \delta)\right\}\right]$$

$$= \frac{mA^2}{2} e^{-2\beta t}\left\{(-2\beta)\left[\omega_N^2 + \beta^2 \cos 2(\omega_s t - \delta) + \beta\omega_s \sin 2(\omega_s t - \delta)\right] + \right.$$

$$\left. + 0 - \beta^2 2\omega_s \sin 2(\omega_s t - \delta) + \beta 2\omega_s^2 \cos 2(\omega_s t - \delta)\right\}$$

$$\boxed{\frac{dE}{dt} = mA^2 e^{-2\beta t}\left\{(\beta\omega_s^2 - \beta^3)\cos 2(\omega_s t - \delta) + - 2\beta^2\omega_s \sin 2(\omega_s t - \delta) - \beta\omega_N^2\right\}}$$

QED!

SEE ATTACHED MATHEMATICA PLOTS